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Asymptotic analysis of rapid forward accelerations of a free-surface pressure

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Abstract. Linear unsteady water waves due to a pressure distribution in the forward motion over the free surface of an inviscid, incompressible, heavy fluid are considered. The motion of pressure system is assumed to be rectilinear and non-uniform starting from rest. The effect of a rapid acceleration is analysed asymptotically. A two-scale expansion is developed for the velocity potential, and estimates for the remainder are established. Hydrodynamic corollaries are derived from the asymptotics obtained. In particular, it is shown how the resistance, which is the horizontal component of the fluid's reaction to the system's motion, depends on the bottom topography varying in the direction of motion.

Key words: pressure distribution, rapid acceleration, two-scale asymptotic expansion, wave resistance

1. Introduction: statement of the problem

Hovercraft travelling forward over calm water are supported by fans exerting a downward pressure on the water surface (see *e.g.* [1]). Ignoring details of the air flow, a hovercraft may be modelled by the forward motion of a prescribed pressure distribution (see [2–4] and references cited therein). Then the waves generated in water are described within the framework of the linearized theory by an initial-boundary-value problem (see [5, Introduction and Part 3]). Under some restrictions on the problem's data, the uniqueness and existence theorems are already established for solutions of some statements of the initial-boundary-value problem in the following works: [6–8] (see also [5, Chapter 9]).

However, the results in [6–8] provide no details of the transient behaviour of the arising water waves. Nevertheless, there are cases in which it is possible to extract information about the propagation of waves in time. In particular, the asymptotic technique developed in [9, 10] (see also [11] and [5, Chapter 10]) allows us to do this for two classes of disturbances. High-frequency disturbances constitute one of these classes which include, in particular, a high-frequency pressure applied to the free surface of water at rest and the high-frequency oscillations of the forward velocity of a submerged body.

The present paper is concerned with an example from the second class of disturbance – the so-called brief disturbance – to which asymptotic analysis is applicable. Our aim is to investigate the effect of the rapid accelerations of a pressure distribution on the resistance to its rectilinear forward motion in the case when the distribution starts its non-uniform motion over the horizontal free surface of water resting over a variable bottom topography and submerged bodies. Using a two-scale expansion for a velocity potential yields an explicit formula that expresses the time-dependence of the resistance during the interval of acceleration.

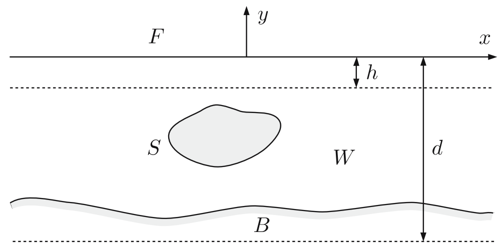

Figure 1. A two-dimensional definition sketch of the geometry.

Furthermore, we show how during the same interval of time the resistance depends on the bottom topography that is variable in the direction of motion.[1]

Let an inviscid, incompressible fluid of density ρ (*e.g.*, water) occupy an infinite domain W that is assumed to be contained in a horizontal layer of constant, possibly infinite, depth $d \in (0, \ +\infty]$ (see Figure 1, where a two-dimensional sketch of the geometry is shown). Cartesian coordinates (x, y, z) are chosen so that $F = \{-\infty < x, z < +\infty, \ \ y = 0\}$ coincides with the mean free surface that bounds W from above, and the y-axis is directed vertically upwards. Along with F, there may be two other parts of the boundary ∂W that are assumed to be rigid: the unbounded sea-bed B and a bounded surface S, which is the union of the wetted boundaries of all immersed bodies; B and S are supposed to be sufficiently smooth surfaces placed at a certain finite distance from F, that is, a layer $\{-\infty < x, z < +\infty, -h < y < 0\}$ of constant depth $h \in (0, d)$ belongs to the water domain W. Hence, generally speaking, $\partial W = F \cup B \cup S$, but either B or S may be empty.

Let the free-surface pressure distribution be given by a smooth function $\mathcal{P}(x, z)$ at the initial moment of time $t = 0$ (Of course, the assumption that $\mathcal{P}$ is smooth is often not satisfied in practical applications, but we impose this assumption because it is essential for justifying the asymptotic formulae derived in the paper). Let $\mathcal{P}$ have a compact support; that is $\mathcal{P}$ vanishes outside a bounded two-dimensional region having the diameter D, and $\mathcal{P} \neq 0$ everywhere inside it. In what follows, it is convenient to apply dimensionless variables using the same notation for the variables and functions already introduced. We take D as the characteristic length, $(D/g)^{1/2}$ as the characteristic time interval, and $\rho D g$ as the characteristic pressure, where g is the acceleration due to gravity. The characteristic scales for other functions are generated by these three. We assume that the pressure distribution does not vary in time, and water waves are generated by the non-uniform forward motion of the distribution. Choosing the x-axis as the direction of the rectilinear motion, we have the following expression for the surface pressure:

$$p(x, z; t) = \mathcal{P}\left(x - \int_0^t \mathcal{V}(\mu) \mathrm{d}\mu, z\right) \quad \text{for } t \geq 0, \tag{1}$$

where $\mathcal{V}(t) \geq 0$ is the time-dependent forward velocity. The latter is supposed to be a continuous function of $t \geq 0$, vanishing at $t = 0$ and depending also on a small parameter ϵ in the following way:

[1]A preliminary version of results presented here was completed almost 20 years ago and announced in the note [12]. The author can only apologize to the readers and editor for the great delay in the publication of the final version of this work.

$$\mathcal{V}(t) = v(t/\epsilon) \geq 0, \quad v(\mu) \to V = \text{const} > 0 \text{ as } \mu \to \infty. \tag{2}$$

Moreover, $Q(\mu) = V - v(\mu)$ must decay at infinity so that

$$\mu^m Q(\mu) \to 0 \text{ as } \mu \to \infty \quad \text{for any } m = 1, 2, \ldots \tag{3}$$

The linearized theory of the irrotational unsteady water waves caused by the moving pressure distribution is formulated (see *e.g.*, [5]) in terms of a velocity potential $\phi(X; t, \epsilon), X = (x, y, z)$. It is natural to assume that ϕ belongs to the class of functions having finite kinetic and potential energy:

$$\int_W |\nabla\phi|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z + \int_F \eta^2 \mathrm{d}x\mathrm{d}z < \infty. \tag{4}$$

Here $\nabla = (\partial x, \partial y, \partial z)$ is the gradient operator and η denotes the free-surface elevation linked to ϕ and p by the linearized kinematic condition on the free surface:

$$\eta(x, z; t) = -[\partial_t \phi(x, 0, z; t) + p(x, z; t)]. \tag{5}$$

The continuity equation for the velocity field implies that ϕ satisfies Laplace's equation

$$\nabla^2 \phi = 0 \text{ in } W \text{ for } t \geq 0. \tag{6}$$

There is no flow through any rigid surface and so

$$\partial_n \phi = 0 \text{ on } B \cup S \text{ for } t \geq 0, \tag{7}$$

where ∂_n indicates differentiation with respect to a unit normal directed into W. The linearized Bernoulli equation and (5) combine to give another free-surface condition:

$$\partial_t^2 \phi + \partial_y \phi = -\partial_t p \text{ on } F \text{ for } t \geq 0. \tag{8}$$

Equation (6–8) are complemented by the following two initial conditions:

$$\phi(x, 0, z; 0) = 0, \tag{9}$$

$$\partial_t \phi(x, 0, z; 0) = -\mathcal{P}(x, z). \tag{10}$$

The meaning of (10) follows from (5) and (1) and expresses the fact that the free surface is horizontal at $t = 0$. According to (9), (7), (6), and (4), we have that $\phi(X; 0, \epsilon)$ vanishes identically in W, which means that there is no initial motion in the water domain.

Our aim is construct an asymptotic expansion for ϕ valid as $\epsilon \to 0$. In order to understand what the assumption $\epsilon \ll 1$ means, one has to consider a velocity $v(\mu)$ that is equal to V identically for $\mu \geq 1$, in which case the velocity varies only during the initial time interval that is short in comparison with the characteristic time interval $(D/g)^{1/2}$. Thus the pressure distribution accelerates rapidly during an initial interval of its motion and then moves forward uniformly. In the case when the general condition (3) holds, the velocity tends to V faster than any power of t/ϵ.

2. Formal asymptotic expansion

The forward velocity defined by (2) involves the so-called 'rapid' time $\tau = t/\epsilon$, and the right-hand-side term in (8) with p given by (1) can be written in the form:

$$-p_t = v(\tau)\partial_x \mathcal{P}(x - Vt + \epsilon\alpha(\tau), z), \quad \text{where } \alpha(\tau) = \int_0^\tau [V - v(\mu)]\mathrm{d}\mu. \tag{11}$$

Here the dependence on ϵ is expressed explicitly. In order to apply the technique of two-scale asymptotic expansions for studying ϕ and deriving the hydrodynamic corollaries, let us expand (11) into a sum of two series in powers of ϵ so that each series depends only on a single time scale τ or t.

First we apply Taylor's formula to (11) and obtain

$$\sum_{m=0}^{\infty} \frac{\epsilon^m}{m!}(v(\tau)[\alpha(\tau)]^m - V[\alpha(\infty)]^m)\partial_x^{m+1}\mathcal{P}(x - Vt, z) + V\sum_{m=0}^{\infty}\frac{\epsilon^m}{m!}[\alpha(\infty)]^m\partial_x^{m+1}\mathcal{P}(x - Vt, z). \tag{12}$$

Here the original expansion is split into two sums in order to apply again Taylor's formula to $\partial_x^{m+1}\mathcal{P}(x - \epsilon V_\tau, z)$ in the first sum. After doing this and rearranging the order of summation, we get

$$\sum_{m=0}^{\infty}\epsilon^m\partial_x^{m+1}\mathcal{P}(x, z)\sum_{k=0}^{m}\frac{(-V\tau)^{m-k}}{k!(m-k)!}\left(v(\tau)[\alpha(\tau)]^k - V[\alpha(\infty)]^k\right) = \sum_{m=0}^{\infty}\frac{\epsilon^m}{m!}\beta_m(\tau)\partial_x^{m+1}\mathcal{P}(x, z), \tag{13}$$

where

$$\beta_m(\tau) = v(\tau)[\alpha(\tau) - V\tau]^m - V[\alpha(\infty) - V\tau]^m, \quad m = 0, 1, \ldots,$$

arises when the binomial formula is applied in the second sum in the left-hand side. Combining (12) with (13), we arrive at the required expansion for (11) or, equivalently, for the right-hand side term in (8):

$$-\partial_t p = \sum_{m=0}^{\infty}\frac{\epsilon^m}{m!}\left\{\beta_m(\tau)\partial_x^{m+1}\mathcal{P}(x, z) + V[\alpha(\infty)]^m\partial_x^{m+1}\mathcal{P}(x - Vt, z)\right\}. \tag{14}$$

Here the first term in braces depends only on τ, whereas the second depends only on t. It is easy to check that

$$\beta_m(0) = -[\alpha(\infty)]^m \text{ and } \beta_m(\tau) \to 0 \text{ as } \tau \to \infty.$$

Let us seeks the velocity potential as the two-time scaled asymptotic series

$$\phi(X; t, \epsilon) = \sum_{m=0}^{\infty}\epsilon^m[\varphi_m(X; \tau) + \psi_m(X; t)], \tag{15}$$

whose form is similar to the form of the series in the right-hand side of (14). Here $\varphi_m(X; \tau)$ is assumed to tend to zero as $\tau \to \infty, m = 0, 1, \ldots$. These functions must also decay as $|X|^2 = x^2 + y^2 + z^2 \to \infty$, so that condition (4) holds for them. The latter property must be true for $\psi_m(X; t)$ as well.

Now we apply a standard asymptotic procedure in order to obtain boundary-value problems for φ_m and ψ_m. After substituting (15) and (14) in (6–10), we equate coefficients at each power of ϵ. Moreover, coefficients depending on τ and t are equated separately. Thus we get the following equations for φ_m holding for $\tau \geq 0$:

$$\nabla^2 \varphi_m = 0 \text{ in } W, \text{ and } \partial_n \varphi_m = 0 \text{ on } B \cup S, \quad m = 0, 1, \dots; \tag{16}$$

$$\partial_\tau^2 \varphi_m = 0 \text{ on } F, \quad m = 0, 1; \tag{17}$$

$$\partial_\tau^2 \varphi_m + \partial_y \varphi_{m-2} = \beta_{m-2}(\tau) \partial_x^{m-1} \mathcal{P}(x, z) \text{ on } F, \quad m = 2, 3, \dots. \tag{18}$$

For $\psi_m, m = 0, 1, \dots,$ we arrive at the following equations valid for $t \geq 0$:

$$\nabla^2 \psi_m = 0 \text{ in } W, \text{ and } \partial_n \psi_m = 0 \text{ on } B \cup S, \tag{19}$$

$$\partial_t^2 \psi_m + \partial_y \psi_m = V[\alpha(\infty)]^m \partial_x^{m+1} \mathcal{P}(x - Vt, z) \text{ on } F. \tag{20}$$

In addition, the following initial relations must hold:

$$\psi_m(x, 0, z; 0) = -\varphi_m(x, 0, z; 0), \quad m = 0, 1, \dots; \tag{21}$$

$$\partial_t \psi_0(x, 0, z; 0) = -[\partial_\tau \varphi_1(x, 0, z; 0) + \mathcal{P}(x, z)]; \tag{22}$$

$$\partial_t \psi_m(x, 0, z; 0) = -\partial_\tau \varphi_{m+1}(x, 0, z; 0), \quad m = 1, 2, \dots. \tag{23}$$

Integrating (17) under the condition that φ_m decays for large τ, we get

$$\varphi_m(x, 0, z; \tau) = 0 \text{ for } \tau \geq 0 \text{ and } m = 0, 1.$$

Thus, (16) and (4) imply that $\varphi_0(X; \tau)$ and $\varphi_1(X; \tau)$ vanish identically for $0 \leq \tau < +\infty$ and $X \in \overline{W} (\overline{W} = W \cup \partial W$ is the closure of $W)$. Hence (18) reduces to

$$\partial_\tau^2 \varphi_m = \beta_{m-2}(\tau) \partial_x^{m-1} \mathcal{P}(x, z) \text{ on } F \text{ for } m = 2, 3.$$

From here we obtain that

$$\varphi_m(x, 0, z; \tau) = \partial_x^{m-1} \mathcal{P}(x, z) \int_\tau^\infty (\mu - \tau) \beta_{m-2}(\mu) \mathrm{d}\mu \quad \text{for } m = 2, 3, \tag{24}$$

because φ_m decays as $\tau \to \infty$. Solving the boundary-value problem (16) and (24), which depends on the parameter τ, in the class defined by (4), one determines $\varphi_2(X; \tau)$ and $\varphi_3(X; \tau)$ uniquely for $X \in \overline{W}$ and $0 \leq \tau < +\infty$. Continuing the iterative procedure, we arrive at the following result:

$$\varphi_m(X; \tau) = \sum_{k=1}^{[m/2]} u_{mk}(X) \frac{1}{(2k-1)!(m-2k)!} \int_\tau^\infty (\mu - \tau)^{2k-1} \beta_{m-2k}(\mu) \mathrm{d}\mu \quad \text{for } m = 2, 3, \dots. \tag{25}$$

Here $[s]$ denotes the integer part of $s \in (-\infty, +\infty)$, and u_{mk} must be determined from the following boundary-value problem:

$$\nabla^2 u_{mk} = 0 \text{ in } W, \quad \partial_u u_{mk} = 0 \text{ on } B \cup S, \tag{26}$$

$$u_{mk} = \begin{cases} \partial_x^{m-1} \mathcal{P}(x, z) & \text{for } k = 1. \\ -\partial_y u_{m-2,k-1} & \text{for } k = 2, 3, \dots \quad \text{on } F, \end{cases} \tag{27}$$

which is uniquely solvable under condition (4). It can be verified by direct calculation that the recurrent condition (18) holds for the functions defined by (25) with u_{mk} satisfying (27). Equations (26) follow from (16).

Now the right-hand-side terms in the initial condition (21–23) are found. This allows us to solve the sequence of initial-boundary-value problems (19–23) that defines ψ_m for $m=0,1,\ldots$.

Let us summarise the results of the present section. We developed the following *algorithm for finding terms in the asymptotic expansion* (15). *Solutions to the recurrent sequence of time-independent boundary-value problems* (26), (27) *must be found first, and then* (25) *determines* $\varphi_m, m=2,3,\ldots,$ *whereas* φ_0 *and* φ_1 *vanish identically. When all* φ_m *are defined, they provide the initial data for the sequence of the initial-boundary-value problems* (19–23), $m=0,1,\ldots$. *Solving the latter problems one finds* ψ_m, *thus completing the construction of the asymptotic expansion* (15).

To conclude this section, we note that, if the water is of constant depth, no submerged bodies are present, then it is possible to integrate explicitly the sequence of problems for φ_m and ψ_m. This can be performed in the same way as in [5, Section 10.1.3].

3. Justification of the asymptotics

To justify the asymptotic formula (15) we have to derive an estimate of the following remainder term

$$r_N(X;t,\epsilon)=\phi(X;t,\epsilon)-\sum_{m=0}^{N}\epsilon^m[\varphi_m(X;\tau)+\psi_m(X;t)].$$

Since φ_2 is the first non-trivial function in the sequences $\{\varphi_m\}_{m=0}^{\infty}$ obtained in Section 2, we assume that $N\geq 2$. Substituting r_N and (11) in problem (6–10) and using (16–23), one can verify directly that r_N must satisfy the following initial-boundary-value problem:

$$\nabla^2 r_N=0 \text{ in } W, \quad \partial_n r_N=0, \text{ on } B\cup S \quad \text{for } t\geq 0, \tag{28}$$

$$\partial_t^2 r_N+\partial_y r_N=f_N(x,z;t,\epsilon) \text{ on } F \text{ for } t\geq 0, \tag{29}$$

$$r_N(x,0,z;0,\epsilon)=\partial_t r_N(x,0,z;0,\epsilon)=0. \tag{30}$$

where

$$\begin{aligned} f_N(x,z;t,\epsilon)=v(\tau)\partial_x\mathcal{P}(x-Vt+\epsilon\alpha(\tau),z)-\\ -\sum_{m=0}^{N-2}\frac{\epsilon^m}{m!}\Big[\beta_m(\tau)\partial_x^{m+1}\mathcal{P}(x,z)+[\alpha(\infty)]^m\partial_x^{m+1}\mathcal{P}(x-Vt,z)\Big]-\\ -\sum_{m=N-1}^{N}\frac{\epsilon^m}{m!}\Big[\partial_y\varphi_m+[\alpha(\infty)]^m\partial_x^{m+1}\mathcal{P}(x-Vt,z)\Big]. \end{aligned} \tag{31}$$

By re-expanding the right-hand side here in the same way as was done for (11), it can be shown that (31) takes the form:

$$f_N(x,z;t,\epsilon)=\frac{\epsilon^{N-1}}{(N-1)!}\left(v(\tau)[\alpha(\tau)]^{N-1}\int_0^1 \partial_x^N\mathcal{P}(x+\epsilon\alpha(\tau)\mu,z)(1-\mu)^{N-1}\mathrm{d}\mu+\right.$$

$$+\left[V\beta_{N-1}(\tau)+v(\sigma)[\alpha(\sigma)]^{N-1}|_{\sigma=\tau}^{\sigma=\infty}\right]\int_0^1 \partial_x^N\mathcal{P}(x-Vt\mu,z)(1-\mu)^{N-1}\mathrm{d}\mu-$$

$$\left.-\left[\partial_y\varphi_{N-1}(x,0,z;\tau)+[\alpha(\infty)]^{N-1}\partial_x^N\mathcal{P}(x-Vt,z)\right]\right)-$$

$$-\frac{\epsilon^N}{N!}\left[\partial_y\varphi_N(x,0,z;\tau)+[\alpha(\infty)]^N\partial_x^{N+1}\mathcal{P}(x-Vt,z)\right]. \tag{32}$$

It is shown in [7] (see also [5, Chapter 9]) that the trace on F of a solution to problem (28–30) can be expressed as follows:

$$r_N(x,0,z;t,\epsilon)=\int_0^t K^{-1/2}\sin\left((t-\mu)K^{1/2}\right)f_N(x,z;\mu,\epsilon)\mathrm{d}\mu. \tag{33}$$

Here K is the so-called Dirichlet-Neumann operator that maps $\varphi(x,z)$ belonging to the Sobolev space $H^\ell(F)$, $-\infty<\ell<+\infty$, into $(K\varphi)(x,z)=u_y(x,0,z)$, where u solves the following boundary-value problem

$$\nabla^2 u=0 \text{ in } W,\quad \partial_n u=0 \text{ on } B\cup S,\quad u=\varphi \text{ on } F,$$

It is well-known that K satisfies the following estimate:

$$\|K\varphi\|_{\ell-1}\le C_\ell\|\varphi\|_\ell. \tag{34}$$

For estimating r_N, it is necessary to split (33) into a sum defined by the right-hand side of (32) and then estimate each of the six terms in the same way as in [5, Chapter 10]. (The crucial point is to use (34).) Omitting the details of this estimation procedure, we formulate the final result:

Let $\mathcal{P}\in H^{N+2}(F)$, then the following estimate

$$\|r_N\|_{1/2}\le C(N)\epsilon^{N+1}t\|\mathcal{P}\|_{N+2}$$

holds, thus justifying the asymptotic formula (15)*. Here $\|\cdot\|_\ell$ denotes the norm in $H^\ell(F)$.*

4. Hydrodynamic corollaries uniform in time

The aim of the present section and of the next one is to derive asymptotic formulae for the wave resistance. In the present section we deal with formulae which are true for any finite subintervals of $t\ge 0$. Other formulae valid for $t=O(\epsilon)$ are considered in Section 5.

Since the elevation of free surface η is involved in the calculation of the resistance (that is, the horizontal component of the reaction of water to the forward motion of the pressure distribution), we begin with developing the asymptotics for η. According to results obtained in Section 2, the principal terms in the asymptotics of $\phi(X;t,\epsilon)$ and $\partial_t\phi(X;t,\epsilon)$ are $\psi_0(X;t)$ and $\partial_t\psi_0(X;t)$, respectively. Applying the procedure used for obtaining (14), we get that the principal term in the asymptotics of $p(x,z;t,\epsilon)$ is equal to $\mathcal{P}(x-Vt,z)$. Substituting these asymptotics in (5), we get

$$\eta(x,z;t,\epsilon)=-[\partial_t\psi_0(x,0,z;t)+\mathcal{P}(x-Vt,z)]+O(\epsilon), \tag{35}$$

which shows that, if the pressure system either instantly starts the forward motion at the limit speed V or approaches the same speed during a time interval $O(\epsilon)$, then on any finite subinterval of $t \geq 0$ the free-surface elevation is the same up to a term $O(\epsilon)$. The asymptotic formula for η including the first-order term has the form:

$$\eta(x,z;t,\epsilon) = -[\partial_t \psi_0(x,0,z;t) + \mathcal{P}(x - Vt, z)] - \\ -\epsilon\,[\partial_t \psi_1(x,0,z;t) + \alpha(\infty)\partial_x \mathcal{P}(x - Vt, z)] + O(\epsilon^2), \tag{36}$$

because the contributions depending on the rapid time τ cancel. This is natural in view of the assumption that the acceleration time scale ϵ is short in comparison with the gravitational time scale $(D/g)^{1/2}$ and the fact that gravity is the force generating water waves (see Section 5, where the gravitational character of waves is explained in more detail for the initial time interval $O(\epsilon)$).

Let us turn to the resistance $R(t,\epsilon)$, which is the horizontal component of the reaction of water to the forward motion of the pressure distribution and is equal to the integrated hydrodynamic pressure force (see *e.g.*, [13, p. 459] for a general formula)

$$R(t,\epsilon) = \int_{y=\eta(x,z;t)} p n_x \mathrm{d}\sigma,$$

where n_x is the x-component of the unit normal n to $y = \eta(x,z;t)$; here and below, n is directed into water and $\mathrm{d}\sigma$ denotes the element of surface area. Since p given by (1) vanishes at infinity, one can integrate over the whole surface $y = \eta(x,z;t)$. Changing variables and substituting (1), we get

$$R(t,\epsilon) = \int_F \mathcal{P}\left(x - \int_0^t v(\mu/\epsilon)\mathrm{d}\mu, z\right) \partial_x \eta(x,z;t,\epsilon)\mathrm{d}x\mathrm{d}z,$$

where now we indicate the dependence of R on ϵ as well. Integrating by parts, we obtain

$$R(t,\epsilon) = -\int_F \eta(x,z;t,\epsilon)\partial_x \mathcal{P}\left(x - \int_0^t v(\mu/\epsilon)\mathrm{d}\mu, z\right)\mathrm{d}x\mathrm{d}z, \tag{37}$$

which is a more convenient formula when η is known. Moreover, using (5) we obtain another representation that involves only the velocity potential:

$$R(t,\epsilon) = \int_F \partial_t \phi(x,0,z;t,\epsilon)\partial_x \mathcal{P}\left(x - \int_0^t v(\mu/\epsilon)\mathrm{d}\mu, z\right)\mathrm{d}x\mathrm{d}z. \tag{38}$$

Here we have taken into account that the second term

$$\mathcal{P}\left(x - \int_0^t v(\mu/\epsilon)\mathrm{d}\mu, z\right)\partial_x \mathcal{P}\left(x - \int_0^t v(\mu/\epsilon)\mathrm{d}\mu, z\right)$$

integrates to zero over F because $\mathcal{P}$ vanishes at infinity.

Using the principal terms in the asymptotics of both factors in the right-hand side of (38), we get

$$R(t,\epsilon) = \int_F \partial_t \psi_0(x,0,z;t)\partial_x \mathcal{P}(x - Vt, z)\mathrm{d}x\mathrm{d}z + O(\epsilon), \tag{39}$$

and, like formula (35), this asymptotics holds on any finite time interval. Again, the principal term in (39) corresponds to the pressure distribution instantly starting its motion at the speed V.

Using the terms of the zero and first order, we obtain from (38)

$$
\begin{aligned}
R(t,\epsilon) = &\int_F \partial_t \psi_0(X_0;t)\partial_x \mathcal{P}(x-Vt,z)\mathrm{d}x\mathrm{d}z + \\
&+\epsilon \int_F \Big[\alpha(\infty)\partial_t\psi_0(X_0;t)\partial_x^2\mathcal{P}(x-Vt,z) + \partial_t\psi_1(X_0;t)\partial_x\mathcal{P}(x-Vt,z)\Big]\mathrm{d}x\mathrm{d}z - \\
&-\epsilon[\alpha(\infty)-\alpha(\tau)]\int_F \partial_t\psi_0(X_0;t)\partial_x\mathcal{P}(x,z)\mathrm{d}x\mathrm{d}z + O(\epsilon^2),
\end{aligned}
$$

where $X_0=(x,0,z)$. Since the τ-dependent factor $[\alpha(\infty)-\alpha(\tau)]$ decays rapidly, we apply formula

$$
\partial_t\psi_0(X_0;t) = -\mathcal{P}(x,z) + O(\epsilon), \tag{40}
$$

that follows from (22). This gives

$$
\begin{aligned}
&-\epsilon[\alpha(\infty)-\alpha(\tau)]\int_F \partial_t\psi_0(X_0;t)\partial_x\mathcal{P}(x,z)\mathrm{d}x\mathrm{d}z \\
&\quad = \epsilon[\alpha(\infty)-\alpha(\tau)]\int_F \mathcal{P}(x,z)\partial_x\mathcal{P}(x,z)\mathrm{d}x\mathrm{d}z + O(\epsilon^2) = O(\epsilon^2),
\end{aligned}
$$

because the integral vanishes since $\mathcal{P}$ vanishes at infinity. Therefore, we get

$$
\begin{aligned}
R(t,\epsilon) = &\int_F \partial_t \psi_0(X_0;t)\partial_x \mathcal{P}(x-Vt,z)\mathrm{d}x\mathrm{d}z + \\
&+\epsilon \int_F \Big[\alpha(\infty)\partial_t\psi_0(X_0;t)\partial_x^2\mathcal{P}(x-Vt,z) + \partial_t\psi_1(X_0;t)\partial_x\mathcal{P}(x-Vt,z)\Big]\mathrm{d}x\mathrm{d}z + O(\epsilon^2).
\end{aligned}
$$

Comparing this with (36), we see that the asymptotic formulae for both η and R do not depend on the rapid time τ up to $O(\epsilon^2)$. Furthermore, one can show that the first coefficient depending on τ in the asymptotics of R is the coefficient in $O(\epsilon^3)$.

5. Hydrodynamic corollaries for $t = O(\epsilon)$

Let us analyse formula (35) for $t=O(\epsilon)$. For this purpose we apply (40), which is a consequence of the initial condition (22), and the analogue of (40) for $\mathcal{P}(x-Vt)$. Thus we find that

$$
\eta(x,z;t,\epsilon) = O(\epsilon) \text{ for } t = O(\epsilon).
$$

Hence, more terms must be considered for obtaining the first non-trivial term in the asymptotics of η for $t=O(\epsilon)$. Therefore, we truncate (15) and the series for p by dropping the terms with $m>2$. Substituting the truncated expansion in (5), we get

$$
\begin{aligned}
\eta(x,z;t,\epsilon) = -\sum_{m=0}^{2}\epsilon^m \Big\{&\partial_\tau\varphi_{m+1}(x,0,z;\tau) + \partial_t\psi_m(x,0,z;\tau) + \\
&+\frac{1}{m!}\Big[([\alpha(\tau)-V\tau]^m - [\alpha(\infty)-V\tau]^m)\partial_x^m\mathcal{P}(x,z) + \\
&+[\alpha(\infty)]^m\partial_x^m\mathcal{P}(x-Vt,z)\Big]\Big\} + O(\epsilon^3).
\end{aligned} \tag{41}
$$

Here we have taken into account that φ_0 vanishes identically. More simplifications are possible when $t=O(\epsilon)$. We begin by dropping $\varphi_1\equiv 0$ and re-expanding functions depending on t

in Taylor series. Then, using (18), (20), (22–25), and (27), we obtain after simple, but rather lengthy algebra that cancels the coefficient at ϵ, that the following applies:

$$\eta(x,z;t,\epsilon) = -\frac{\epsilon^2\tau^2}{2}\partial_y\partial_t\psi_0(x,0,z;0) + O(\epsilon^3) \text{ for } t = O(\epsilon). \tag{42}$$

Since $\tau = t/\epsilon$, *the principal term in the asymptotics of the free-surface elevation is proportional to* $t^2/2$ when $t = O(\epsilon)$. Presumably, the gravitational origin of water waves causes the similarity between the principal term in (42) and the dependence on time of the distance of a body falling down under the action of gravity.

To clarify the dependence of η on $\mathcal{P}$ during the initial time interval, let us rewrite (42). Putting $U_0(X) = -\partial_t\psi_0(X;0)$, we see that, by (19) and (22), this function satisfies the following boundary-value problem:

$$\nabla^2 U_0 = 0 \text{ in } W,\ \partial_n U_0 = 0 \text{ on } B \cup S,\ U_0 = \mathcal{P}(x,z) \text{ on } F. \tag{43}$$

(We recall that this boundary-value problem has the unique solution in the class defined by (4).) Now (42) takes the form:

$$\eta(x,z;t,\epsilon) = \frac{t^2}{2}\partial_y U_0(x,0,z) + O(\epsilon^3) \text{ for } t = O(\epsilon). \tag{44}$$

Since $\mathcal{P}$ is smooth and vanishes at infinity, it attains its maximum and minimum on the support of $\mathcal{P}$. To be specific, let this function have its maximum at (x_*, z_*). Then U_0 also has its maximum at $(x_*, 0, z_*)$. This follows from (43), the maximum principle for harmonic functions, and the maximum principle of Hopf (see, *e.g.*, [14, Chapter 2, Section 3]). Indeed, by the first principle U_0 has its maximum on ∂W. Since the second principle implies that the outward normal derivative is positive at the point where the maximum is reached, this point cannot belong to $B \cup S$. Hence $(x_*, 0, z_*)$ is the maximum point of U_0 because of the Dirichlet condition that holds on F. Moreover, by the maximum principle of Hopf, we have that $\partial_y U_0(x_*, 0, z_*) > 0$. The case of the minimum of $\mathcal{P}$ can be considered in the same way. Hence we arrive at the following conclusion:

The asymptotic formula (44) *implies that during the time interval* $t = O(\epsilon)$ *the free-surface elevation is positive (negative) at the point where the pressure distribution attains its maximum (minimum).*

Let us turn to the physical meaning of this result. It says that the point on the horizontal free surface, subjected to the maximum pressure at the initial moment, moves upwards after being released from the action of the maximum pressure because of its forward motion.

When analysing below the resistance of a pressure distribution moving over the horizontal bottom, we need the two-term asymptotic formula for η in addition to (44). The same procedure as above, but starting from (41) with four terms, gives

$$\eta(x,z;t,\epsilon) = \frac{t^2}{2}\partial_y U_0(x,0,z) + \epsilon^3\frac{\chi(\tau)}{2}\partial_y U_1(x,0,z) + O(\epsilon^4) \text{ for } t = O(\epsilon), \tag{45}$$

where

$$\chi(\tau) = \int_0^\tau [V\mu^2 - (V - v(\mu))(\tau-\mu)^2]\,\mathrm{d}\mu, \quad \tau = t/\epsilon, \tag{46}$$

and $U_1(X) = u_{21}(X)$ satisfies the following boundary-value problem (see (26) and (27)):

$$\nabla^2 U_1 = 0 \text{ in } W,\ \partial_n U_1 = 0 \text{ on } B \cup S,\ U_1 = \partial_x\mathcal{P}(x,z) \text{ on } F, \tag{47}$$

which is uniquely solvable in the class defined by (4).

It is obvious that $\chi(0)=0$ and, differentiating (46), we have

$$\chi'(\tau)=V\tau^2-2\int_0^\tau(\tau-\mu)(V-v(\mu))\mathrm{d}\mu=2\int_0^\tau(\tau-\mu)v(\mu)\mathrm{d}\mu\geq 0, \tag{48}$$

because $v\geq 0$. Thus $\chi(\tau)\geq 0$ and, being an increasing function, $\chi(\tau)$ becomes strictly positive for $\tau>0$ as soon as $v(\tau)$ does.

In order to obtain the leading term in the asymptotics of $R(t,\epsilon)$ valid for $t=O(\epsilon)$, we substitute (44) in (37) and expand the x-derivative of the pressure in the same way as in Section 2. Then we arrive at

$$R(t,\epsilon)=-\frac{t^2}{2}\int_F \partial_y U_0(x,0,z)\partial_x \mathcal{P}(x,z)\mathrm{d}x\mathrm{d}z+O(\epsilon^3)\ \text{ for } t=O(\epsilon). \tag{49}$$

The explicit dependence on t is a significant advantage of this formula as compared with (39). Indeed, similarly to (44) the principal term in (49) is proportional to t^2, whereas for revealing the dependence on t in (39) one has to solve the initial-boundary-value problem for ψ_0 and take into account the forward motion of the pressure distribution with velocity V. After transforming (49), we will derive more corollaries concerning the behaviour of $R(t,\epsilon)$ for $t=O(\epsilon)$.

We note that $\partial_x\mathcal{P}(x,z)=\partial_x U_0(x,0,z)$ by the last condition in (43). This and the fact that U_0 satisfies the homogeneous Neumann condition on $B\cup S$ (see (43)) allow us to apply Green's identity to the integral in (49). The result is as follows:

$$R(t,\epsilon)=-\frac{t^2}{2}\int_W \nabla U_0\cdot\nabla\partial_x U_0\mathrm{d}x\mathrm{d}y\mathrm{d}z+O(\epsilon^3).$$

Since $\nabla U_0\cdot\nabla\partial_x U_0=2^{-1}\partial_x|\nabla U_0|^2$, the divergence theorem gives the final asymptotic formula for the resistance:

$$R(t,\epsilon)=\frac{t^2}{4}\int_{B\cup S}|\nabla U_0|^2 n_x\mathrm{d}\sigma+O(\epsilon^3)\ \text{ for } t=O(\epsilon). \tag{50}$$

Here we integrate only over $B\cup S$ because n_x vanishes identically on F. Note that the second x-derivative of $\mathcal{P}$ was used in the formula (41), which is the starting point for deriving formula (50), and so the latter formula cannot be obtained without some smoothness assumption.

Comparing (49) with (50), we see that the advantage of (49) follows from the fact that one has to integrate over a bounded subregion of F, because $\mathcal{P}$ has compact support. If B is flat outside a bounded region, that is, B coincides at infinity with

$$\{-\infty<x,z<+\infty,\ y=-d\},\quad d=\text{const}>0, \tag{51}$$

then n_x has compact support on B, and one also integrates in (50) over a bounded region. (Note that one cannot avoid integration over the unbounded domain W in the equivalent formula involving W.)

Let us turn to hydrodynamic corollaries of (50). Since the first factor in the integrand is positive, the sign of $R(t,\epsilon)$ depends on the sign of n_x. First, let us mention two cases when the leading term in (50) does vanish:

1. There are no submerged bodies and the bottom is flat; that is, $S=\emptyset$ and B is given by (51). Of course, various techniques can be applied in this case; for example, an explicit solution can be obtained in the form of integral transforms which allows us to analyse

this solution by means of asymptotic methods developed for integrals (mainly for integrals with a large parameter; see, *e.g.*, [15] and references cited therein). However, below we use for this particular geometry the same asymptotic method as for the general geometry.

2. Surfaces S and B are symmetric about the (y,z)-plane, in which case $\cos(n,x)$ is an odd function of x. The pressure distribution is either symmetric or antisymmetric about the same vertical plane; that is, $\mathcal{P}(-x,z)=\pm\mathcal{P}(x,z)$, which implies that $|\nabla U_0(x,y,z)|^2$ is an even function of x. Hence the integrand in (50) is odd in x, and so the integral vanishes.

Now let us assume that $S=\emptyset$ and B is a cylindrical surface having its generators parallel to the z-axis; that is, the bottom is defined as follows:

$$B=\{-\infty<x,z<+\infty,\ y=-H(x)\},\quad \text{where } H(x)>0. \tag{52}$$

Therefore, if H is monotonic, then n_x is of fixed sign on B; that is, $n_x\geq 0$ ($n_x\leq 0$) when $-H$ decreases (increases). Hence *when the pressure distribution accelerates down (up) the bottom slope, the principal term in the asymptotic formula* (50) *for the resistance is positive (negative).*

Let us discuss what the latter assertion means in terms of the reaction force acting on the pressure system during its motion. It is natural that $R<0$ when the system accelerates up the bottom slope, which means that this force acts in the direction opposite to the x-axis along which the pressure system moves. However, it is an unexpected result that, when the pressure system accelerates down the bottom slope, the force acts in the direction of motion because the leading term of the resistance is positive.

Let us consider the first case mentioned above in more detail. Since the corresponding conditions imply that the leading term in (50) vanishes, we substitute (45) in (37) and keep two terms in the expansion of $\partial_x\mathcal{P}(x-\epsilon\int_0^\tau v(\mu)\mathrm{d}\mu,z)$ in (37). Then we arrive at the following asymptotics:

$$R(t,\epsilon)=\frac{\epsilon^3}{2}\int_F\left[\left(\tau^2\int_0^\tau v(\mu)\mathrm{d}\mu\right)\partial_x^2\mathcal{P}(x,z)\partial_y U_0-\chi(\tau)\partial_x\mathcal{P}(x,z)\partial_y U_1\right]\mathrm{d}x\mathrm{d}z+O(\epsilon^4),\quad \text{for } t=O(\epsilon). \tag{53}$$

Here the τ-dependent functions are nonnegative by (2) and (48), and they become positive as soon as $v(\tau)$ does. Let us split the integral over F in (53) into a sum of two integrals and transform them in order to show that the leading term is negative. Using (47) and Green's identity (this is possible because condition (4) implies that there is no contribution of the integral over a vertical circular cylinder whose radius tends to infinity), we have for the second integral:

$$\int_F\partial_x\mathcal{P}(x,z)\partial_y U_1\mathrm{d}x\mathrm{d}z=\left[\int_F-\int_B\right]U_1\partial_y U_1\mathrm{d}x\mathrm{d}z=\int_W|\nabla U_1|^2\mathrm{d}x\mathrm{d}y\mathrm{d}z, \tag{54}$$

where we also take into account that $S=\emptyset$. According to the third condition in (47) and the second condition in (43), the first integral over F in (53) can be transformed as follows:

$$\int_F\partial_x^2\mathcal{P}(x,z)\partial_y U_0\mathrm{d}x\mathrm{d}z=\int_F\partial_x U_1\partial_y U_0\mathrm{d}x\mathrm{d}z=\left[\int_F-\int_B\right]\partial_x U_1\partial_y U_0\mathrm{d}x\mathrm{d}z.$$

Now applying Green's formula to the harmonic functions U_0 and $\partial_x U_1$ we get that the last difference of two integrals is equal to

$$\left[\int_F-\int_B\right]U_0\partial_x\partial_y U_1\mathrm{d}x\mathrm{d}z.$$

Integrating by parts here (again (4) allows us to do this), and using the second condition in (47) and the third condition in (43) differentiated with respect to x, we rewrite the last difference as

$$-\left[\int_F - \int_B\right] \partial_x U_0 \partial_y U_1 \mathrm{d}x\mathrm{d}z = -\int_F \partial_x \mathcal{P}(x,z) \partial_y U_1 \mathrm{d}x\mathrm{d}z.$$

Repeating once more the same manipulations, but now the last two conditions in (47) must be used, and applying once more Green's formula for U_1, we get

$$-\int_F \partial_x \mathcal{P}(x,z)\partial_y U_1 \mathrm{d}x\mathrm{d}z = -\left[\int_F - \int_B\right] U_1 \partial_y U_1 \mathrm{d}x\mathrm{d}z = -\int_W |\nabla U_1|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Thus we have obtained that

$$\int_F \partial_x^2 \mathcal{P}(x,z)\partial_y U_0 \mathrm{d}x\mathrm{d}z = -\int_W |\nabla U_1|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z, \tag{55}$$

where in the right-hand side we have the same integral as in (54). Substituting (54) and (55) in (53), we arrive at

$$R(t,\epsilon) = -\frac{\epsilon^3}{2}\left[\chi(\tau) + \tau^2 \int_0^\tau v(\mu)\mathrm{d}\mu\right] \int_W |\nabla U_1|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z + O(\epsilon^4) \text{ for } t = O(\epsilon). \tag{56}$$

We recall that here $S = \emptyset$, B is given by (51), $\tau = t/\epsilon$, and $\chi(\tau)$ is given (46). Taking into account that the τ-dependent coefficient in the square brackets is non-negative (it becomes strictly positive as soon as $v(\tau)$ does), we see that the leading term in (56) is non-positive (strictly negative as soon as $v(\tau) > 0$).

From (50) and (56), we see that *the behaviour of the resistance during a brief acceleration interval of a pressure distributions moving over the free surface allows us to 'recognise' the bottom topography if it is given by* (52). According to (56), *the water layer of constant depth resists to the acceleration, and* $R = O(\epsilon^3)$. On the other hand, (50) *demonstrates that* $R = O(\epsilon^2)$ *if the bottom is inclined to the horizontal, and the leading term is of definite sign that depends on the character of the bottom slope. The water layer resists the acceleration of the pressure distribution 'uphill', whereas in the case of the 'downhill' motion the water layer shows the opposite reaction.*

6. Conclusion

Waves caused by a rapid acceleration of a pressure distribution over the free surface of water have been considered. In the case when the rate of acceleration is characterised by a small parameter ϵ, a two-scale asymptotic expansion has been derived for the corresponding velocity potential. The remainder term obtained by truncation of the expansion was estimated in appropriate function spaces. Some hydrodynamic corollaries have been deduced from the asymptotics of the velocity potential. These corollaries include two types of asymptotic formulae for the resistance $R(t,\epsilon)$, which is the reaction of water to the forward motion of the pressure system. The relative roles of these formulae are as follows:

- The formulae that belong to the first type (see the last two paragraphs of Section 4, in particular, formula (39)) involve functions ψ_0 and ψ_1, which are solutions of some initial-boundary-value problems. Therefore, these formulae are rather difficult for evaluation, but their advantage over the formulae of the second type (see (50) and (56) in Section 5) is that they are valid on any finite time interval, whereas the formulae of the second type express $R(t,\epsilon)$ only during the initial time interval $t = O(\epsilon)$.

- The qualitative meaning of formula (39) is that, up to $O(\epsilon)$, the resistance $R(t,\epsilon)$ is the same as the resistance of the same pressure distribution instantly starting its forward motion at the limit value of speed.
- The main advantage of the formulae of the second type is the explicit dependence on t of the leading term in the asymptotic of $R(t,\epsilon)$, but, as was mentioned above, this is achieved at the expense of the short time range of these formulae.
- Another advantage of formulae (50) and (56) is the form of dependence on the geometry of the water domain W. This dependence involves function U_0 and U_1 which are solutions of the time-independent boundary-value problems. This allows us to study the qualitative behaviour of $R(t,\epsilon)$ for some particular geometries of W. It was found that, during the rapid acceleration of the surface pressure and depending upon the geometry of W, the resistance can act in the direction opposite to the direction of motion as well as in the same direction. Earlier, a similar effect was discovered in [11]. This study was concerned with the problem of the wave-making resistance for a submerged body, moving forward so that its velocity oscillates with high frequency about a mean value. This problem was studied using another asymptotic technique.

Finally, let us turn to the question how small must be ϵ in the asymptotic procedure presented in this paper. In [5, Section 10.1] a similar asymptotic technique was applied to the problem of impulsive surface pressure and an example constructed for comparing the exact solution with its asymptotic approximation. The numerical computations illustrating the latter example show that even $\epsilon = 3/2$ may be considered to be small enough for the principal term of the asymptotic to provide a good approximation to the exact solution.

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